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Colliding plane waves in string theory

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Abstract: We construct colliding plane wave solutions in higher dimensional gravity theory with dilaton and higher form flux, which appears naturally in the low energy theory of string theory. Especially, the role of the junction condition in constructing the solutions is emphasized. Our results not only include the previously known CPW solutions, but also provide a wide class of new solutions that is not known in the literature before. We find that late time curvature singularity is always developed for the solutions we obtained in this paper. This supports the generalized version of Tipler's theorem in higher dimensional supergravity.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Classical Theories of Gravity, Integrable Equations in Physics.

Contents

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1. Introduction

The gravitational colliding plane wave (CPW) was first studied as an exact solution of Einstein equation by Szekeres [1] and Khan and Penrose [2] in their pioneering papers and have received much attention since then, see [3] and the references therein. See also [4] for connection with integrable systems, and [5] for an exposition on the relation of CPW with Backlund transformation and inverse scattering method.

Unlike the collision of waves in electromagnetic theory, gravitational waves can interact nontrivially due to the nonlinear nature of the Einstein equations. One of the intriguing feature of CPW is the inevitable late time scalar curvature singularity which signifies the non-linearity of the theory. The curvature singularities was initially discovered by Khan and Penrose in their original paper [2]. It was then shown [6, 7] to be a general consequence of colliding plane wave spacetime with plane symmetry. For a brief review of this theorem, see, for example, [8]. Similar null-like or spacelike curvature singularities also arise in the big-bang cosmology and in the black hole, which leads to the breakdown of the classical gravity. It is hoped that further study of the CPW may lead to a new understanding of this kind of the curvature singularities.

CPW has appeared in various interesting physical settings. For example it has been argued [9, 10, 11] that the collision of gravitational plane waves could lead to initial conditions of the primordial cosmological perturbations. In [12], similar construction of CPW has lead to the new type of inhomogeneous cosmology known as Gowdy universe. CPW has also been employed as a useful approximation to study Planckian scattering [13]. The implication to the null-like singularity of the Kerr black hole has been discussed in [14]. See also [15, 16] for CPW within the context of string theory .

To obtain an exact solution of the Einstein equation which describes the collision of gravitational plane waves, one first divides the spacetime into 4 regions, namely, past P-region ($u < 0$, $v < 0$), right R-region ($u > 0$, $v < 0$), left L-region ($u < 0$, $v > 0$), and future F-region ($u, v > 0$) which describes respectively, the Minkowski space before the plane waves arrive, the incoming waves from right and left, and the collision region. One then solves the differential equation of motion within the interior of each region. However, the existence of solutions to the differential equations in each region is not guaranteed to describe the collision of waves in the whole spacetime. As a physical solution, it is necessary that these solutions can be joined to each other in a "physical" way at the (null) junctions. The physical conditions can be translated into conditions on the metric and are called the junction conditions.

One of the generally accepted junction conditions in general relativity is the Lichnerowicz condition which requires that the metric is C^1 and piecewise C^2. This condition guarantees that the curvature tensor is piecewise continuous (P.C.) and nowhere suffers anything more than a 'shock' discontinuity, i.e. no delta function singularity. However, the Lichnerowicz condition was found to be violated for the collision of the Khan-Penrose impulsive waves [2]. This was also realized by Bell and Szekeres when they tried to construct the CPW with an electromagnetic (EM) field shock wave profile [17]. In the setting they considered, a EM potential is included. Therefore a P.C. stress tensor is allowed; and so is the same for the Ricci tensor. Bell and Szekeres thus proposed to consider piecewise C^1 metric which admits a P.C. Ricci tensor. Since for a general piecewise C^1 metric, the Ricci tensor contains a δ-function singularity, it is necessary to impose further conditions on the piecewise C^1 metric such that the δ-function singularity does not appear. These are precisely the O'Brien-Synge (OS) junction conditions [18]. See our appendix B for a detailed exposition.

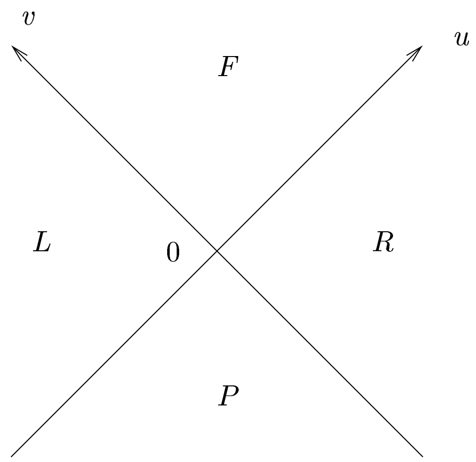

Figure 1: Spacetime in colliding plane wave problem.

Apart from the above-mentioned δ-function singularity, it is also possible that the curvature invariants[1] R and R_2 blow up at the junction. This should be avoided for a physical solution and generally requires additional condition besides the Lichnerowicz/O'Brien-Synge junction conditions. Our section 3 contains a general discussion on this point.

Most of the discussions of CPW have been limited to the 4-dimensional gravity with or without a EM field. Dilatonic gravitational CPW with a EM field has been considered [19]. Higher dimensional generalization of the Bell-Szekeres solution of Maxwell-Einstein gravity has also been attempted recently [20]. As we will demonstrate in section 4, the later solutions however violate the OS junction conditions and are thus not acceptable. Recently, Gutperle and Pioline [21] has tried to construct the CPW in 10-dimensional IIB string theory with the self-dual form flux; for latter convenience, we call the general CPW with form flux the "flux-CPW". However, they found that the curvature invariants blow up at the junction so that the solutions cannot be used to describe the flux-CPW. The goal of this paper is to construct regularly patched flux-CPW solution in string theory, and we find that the key ingredient is to turn on the dilaton field.

In this paper, we construct the flux-CPW solutions in higher dimensional dilaton gravity, which includes the usual 10-dimensional II supergravity with either RR or NS form fluxes. We find that by allowing a dilaton, the pole-like singularity at the junctions in the curvature invariants can be avoided and we obtained the higher dimensional flux-CPW solutions with dilaton. Moreover, by adopting a new form of ansatz which is different from the typical ones used by Bell-Szekeres, we obtain a new class of flux-CPW solutions whose form has never been considered in the literatures.

The organization of the paper is as follows. In section 2, we set up and solve the differential equations in each of the four regions. We find two type of solutions. One of which follows from an ansatz whose form has never been considered in the literature. In section 3, we analyze carefully the necessary (as well as some additional, but uncompulsory) junction conditions that have to be imposed on the metric. The discussions in this section are general without referring to any particular solution. In section 4, we impose these junction conditions on the solutions we find in section 3 and obtain physically acceptable flux-CPW solutions. We also discuss the properties of these solutions. In particular, we find that a curvature singularity is always developed in the future. We conclude with a few discussions in section 5. Appendix A contains some formulae for the Ricci and Riemann tensor for the metric ansatz used for discussing CPW. Appendix B contains a discussion of the δ-function singularity in $R_{\mu\nu\alpha\beta}, R_4, R_{\mu\nu}$ for a general piecewise C^1 metric. We show explicitly that the absence of δ-function singularity in the Ricci tensor is precisely the OS junction conditions. In appendix C we give solution for the pure gravitation CPW and dilatonic CPW without flux. Appendix D gives some details about the boundary behavior for the CPW solutions we obtained. Finally in appendix E, we give details on how the CPW solutions written in the Rosen coordinates appear in the Brinkmann coordinate. In particular we show that the singular structure of R_4 at the junction is the same as the wavefront profile of the wave when written in the Brinkmann coordinates.

[1]Here $R := g^{\mu\nu}R_{\mu\nu}$, $R_2 := R^{\mu\nu}R_{\mu\nu}$, $R_4 := R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ etc.

2. Equation of motion and their solution

Consider the following action with the metric, dilaton and a $(n+1)$-form field strength in D-spacetime dimensions

$$S = \int d^D x\sqrt{-g}\Big(R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2(n+1)!}e^{a\phi}F^2\Big)\,, \tag{2.1}$$

where $F^2 := F_{\mu_1\cdots\mu_{n+1}}F^{\mu_1\cdots\mu_{n+1}}$ and a is the dilaton coupling constant. This action describes a sector of the low energy effective action of string theory in the Einstein frame. The values of a depends on which string theory we are working with and on the nature of the $(n+1)$-form field strength. The equation of motions are given by

$$R_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \frac{1}{2n!}e^{a\phi}\Big(F_{\mu\mu_1\cdots\mu_n}F_\nu{}^{\mu_1\cdots\mu_n} - \frac{n}{(n+1)(n+m)}g_{\mu\nu}F^2\Big),\tag{2.2}$$

$$\partial_\mu(\sqrt{-g}e^{a\phi}F^{\mu\mu_1\cdots\mu_n}) = 0\,, \tag{2.3}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = \frac{a}{4(n+1)!}e^{a\phi}F^2\,. \tag{2.4}$$

2.1 CPW ansatz for the equation of motion

Consider the following ansatz for the metric

$$ds^2 = 2e^{-M}dudv + e^A\sum_{i=1}^{n}dx_i^2 + e^B\sum_{j=1}^{m}dy_j^2\,, \tag{2.5}$$

where $D = 2+n+m$; and the nonzero components of the $(n+1)$-form flux,

$$F_{ux_1\cdots x_n} = C_u\,, \qquad F_{vx_1\cdots x_n} = C_v\,. \tag{2.6}$$

We will take the functions M, A, B, C as well as the dilaton field ϕ to be function of u, v. The Einstein equations take the form:

$$nA_{uu} + mB_{uu} + nM_uA_u + mM_uB_u + \frac{1}{2}(nA_u^2 + mB_u^2) = -2\phi_u^2 - e^{a\phi-nA}C_u^2\,, \tag{2.7}$$

$$nA_{vv} + mB_{vv} + nM_vA_v + mM_vB_v + \frac{1}{2}(nA_v^2 + mB_v^2) = -2\phi_v^2 - e^{a\phi-nA}C_v^2\,, \tag{2.8}$$

$$-M_{uv} + \frac{n}{2}A_{uv} + \frac{m}{2}B_{uv} + \frac{1}{4}(nA_uA_v + mB_uB_v) = -\phi_u\phi_v + \frac{1}{2}\frac{n-m}{n+m}e^{a\phi-nA}C_uC_v\,, \tag{2.9}$$

$$2A_{uv} + nA_uA_v + \frac{m}{2}(A_uB_v + A_vB_u) = -\frac{2m}{n+m}e^{a\phi-nA}C_uC_v\,, \tag{2.10}$$

$$2B_{uv} + mB_uB_v + \frac{n}{2}(A_uB_v + A_vB_u) = \frac{2n}{n+m}e^{a\phi-nA}C_uC_v\,. \tag{2.11}$$

These follow from the $R_{uu}, R_{vv}, R_{uv}, R_{xx}, R_{yy}$-equations (in this order). The equation of motion for dilaton and n-form potential are given by

$$2C_{uv} + \Big(a\phi - \frac{1}{2}(nA - mB)\Big)_u C_v + \Big(a\phi - \frac{1}{2}(nA - mB)\Big)_v C_u = 0\,, \tag{2.12}$$

$$\phi_{uv} + \frac{1}{4}(nA + mB)_u\phi_v + \frac{1}{4}(nA + mB)_v\phi_u = \frac{a}{4}e^{a\phi-nA}C_uC_v\,. \tag{2.13}$$

Here we have abbreviated the derivatives by a subscript, e.g. $A_u = \partial_u A$. Note that equation (2.9) is a consequences of the other equations and is not an independent equation. So in the following we will not write it anymore.

It is convenient to define

$$U = \frac{1}{2}(nA + mB)\,, \qquad V = \frac{1}{2}(nA - mB)\,. \tag{2.14}$$

The equations (2.7), (2.8), (2.10), (2.11) become

$$\begin{aligned}
U_{uu} + M_u U_u + \frac{m+n}{4mn}(U_u^2 + V_u^2) + \frac{m-n}{2mn}U_u V_u &= -\phi_u^2 - \frac{1}{2}e^{a\phi - nA}C_u^2\,, &(2.15)\\
U_{vv} + M_v U_v + \frac{m+n}{4mn}(U_v^2 + V_v^2) + \frac{m-n}{2mn}U_v V_v &= -\phi_v^2 - \frac{1}{2}e^{a\phi - nA}C_v^2\,, &(2.16)\\
U_{uv} + U_u U_v &= 0\,, &(2.17)\\
V_{uv} + \frac{1}{2}(U_u V_v + U_v V_u) &= -\frac{mn}{m+n}e^{a\phi - nA}C_u C_v\,, &(2.18)
\end{aligned}$$

Equation (2.17) says that e^U is a free field and the general solution to it is

$$U = \log(f(u) + g(v)), \tag{2.19}$$

where f, g are arbitrary functions. We will choose them to be monotonic functions. One may treat f, g as coordinates alternative to (u, v).

It is convenient to change from the pair of variables V, ϕ to E, X defined as follows:

$$E = V - a\phi\,, \qquad X = \phi + \delta a V\,, \tag{2.20}$$

where the constant δ is chosen to be

$$\delta := \frac{n+m}{4nm} \leq \frac{1}{2}\,. \tag{2.21}$$

With this choice and some linear combinations, the equations (2.12), (2.13) and (2.18) take the simple form in terms of the (f, g)-coordinates

$$\begin{aligned}
(f+g)X_{fg} + \frac{1}{2}(X_f + X_g) &= 0\,. &(2.22)\\
2C_{fg} - E_f C_g - E_g C_f &= 0\,, &(2.23)\\
(f+g)E_{fg} + \frac{1}{2}(E_f + E_g) &= -\frac{\alpha}{4\delta}e^{-E}C_f C_g\,, &(2.24)
\end{aligned}$$

where

$$\alpha := 1 + a^2\delta\,. \tag{2.25}$$

The equations (2.15) and (2.16) can be integrated to give M. In terms of the (f, g)-coordinates, they can be written as

$$\begin{aligned}
S_f + \frac{1}{2}e^{-E}C_f^2 + \frac{\delta}{\alpha}(f+g)E_f^2 + \frac{1}{\alpha}(f+g)X_f^2 &= 0\,, &(2.26)\\
S_g + \frac{1}{2}e^{-E}C_g^2 + \frac{\delta}{\alpha}(f+g)E_g^2 + \frac{1}{\alpha}(f+g)X_g^2 &= 0\,, &(2.27)
\end{aligned}$$

where we have defined

$$S = M - (1-\delta)\log(f+g) + \log(f_u g_v) + \eta V\,, \tag{2.28}$$

and

$$\eta := \frac{m-n}{2mn}\,. \tag{2.29}$$

We remark that $X = 0$ when there is no dilaton. In this case, $E = V$, $\alpha = 1$ and (2.26), (2.27) reduce to the equation of non-dilatonic gravity with form flux. Therefore the last term in (2.26), (2.27) can be identified as a contribution of the dilaton field to S, i.e. to the metric component e^{-M}. As we will show in section 4, this contribution plays an essential role to allow for physically acceptable CPW solution, i.e. one that satisfies the junction conditions as spelled out in section 3.

The inverse relation of (2.20) is

$$V = \frac{1}{\alpha}(E + aX)\,, \qquad \phi = \frac{1}{\alpha}(X - a\delta E)\,. \tag{2.30}$$

Summarizing, the equation of motion for our system is given by the equations (2.22)–(2.27). Our goal is to solve them for the variables (S, E, X, C). Then using (2.30) (2.28), we can solve for (M, A, B, C, ϕ). In the next subsection, we will give two particular families of solution to the differential equations.

2.2 Two flux-CPW solutions to the equation of motion

The case without form flux is easy to solve and the solutions are given in the appendix C. In the following we will assume that potential C is nonzero. To solve for the most general solution for the above set of coupled differential equations is very difficult. In the following we will give two different solutions using two different form of ansatz.

We first consider the X-equation (2.22). We note that it takes the same form as in the standard pure gravitational plane wave collision [1, 3], and it can be solved by the Khan-Penrose-Szekeres solution:

$$X = \kappa_1 \log\frac{w-p}{w+p} + \kappa_2 \log\frac{r-q}{r+q}\,, \tag{2.31}$$

where κ_1 and κ_2 are integration constants and

$$p := \sqrt{\frac{1}{2} - f}\,, \qquad q := \sqrt{\frac{1}{2} - g}\,, \qquad r := \sqrt{\frac{1}{2} + f}\,, \qquad w := \sqrt{\frac{1}{2} + g}\,. \tag{2.32}$$

Solution I: $(pqrw)$-type. Our solution I is given by the following ansatz for E and C:

$$E = \log\frac{rw + pq}{rw - pq}\,, \qquad C = \gamma(pw - rq)\,, \tag{2.33}$$

which solves (2.23) automatically and from (2.24) we find that

$$\gamma^2 = \frac{8\delta}{\alpha}\,. \tag{2.34}$$

After integrating (2.26) and (2.27) with X given by (2.31) to get

$$\begin{aligned} S = {} & b_1 \log(1-2f)(1+2g) + b_2 \log(1+2f)(1-2g) + (b_3 - 1 + \delta)\log(f+g) + \\ & + \frac{2\kappa_1\kappa_2}{\alpha}\log\left(\frac{1}{2} + 2fg + 2pqrw\right), \end{aligned} \tag{2.35}$$

where

$$b_1 = \frac{\kappa_1^2+\delta}{\alpha}, \qquad b_2 = \frac{\kappa_2^2+\delta}{\alpha}, \qquad b_3 = 1 - \delta - \frac{\delta + (\kappa_1+\kappa_2)^2}{\alpha}, \tag{2.36}$$

and using (2.28) we find that

$$\begin{aligned} e^{-M} = {} & f_u g_v [(1-2f)(1+2g)]^{-b_1}[(1+2f)(1-2g)]^{-b_2}(f+g)^{-b_3} \times \\ & \times \left[\frac{1}{2} + 2fg + 2pqrw\right]^{-\frac{2\kappa_1\kappa_2}{\alpha}} \left(\frac{rw+pq}{rw-pq}\right)^{\frac{\eta}{\alpha}} \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{a\eta}{\alpha}}. \end{aligned} \tag{2.37}$$

The other components of the metric are given by

$$e^{nA} = (f+g)\left(\frac{rw+pq}{rw-pq}\right)^{\frac{1}{\alpha}} \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{a}{\alpha}}, \tag{2.38}$$

$$e^{mB} = (f+g)\left(\frac{rw+pq}{rw-pq}\right)^{-\frac{1}{\alpha}} \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{-\frac{a}{\alpha}} \tag{2.39}$$

and the dilaton field is given by

$$e^{\phi} = \left(\frac{rw+pq}{rw-pq}\right)^{-\frac{\delta a}{\alpha}} \left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{1}{\alpha}}. \tag{2.40}$$

The above (2.31), (2.33)–(2.40) solve the equations of motion for the F-region and represents a two-parameters family of solutions depending on the constants κ_1 and κ_2. They still solve the EOM for the L-region, or for the R-region, or for the P-region if one do the following replacements:

$$f(u) = f_0\,, \quad f_u(1-2f)^{-b_1}|_{f=f_0} = -1 \qquad \text{for } u<0\,, \tag{2.41}$$

$$g(v) = g_0\,, \quad g_v(1-2g)^{-b_2}|_{g=g_0} = -1 \qquad \text{for } v<0\,, \tag{2.42}$$

for some constants f_0, g_0. In the next section, we will discuss the patching of the solutions of the different regions. This allow us to fix the values of

$$f_0 = g_0 = \frac{1}{2} \tag{2.43}$$

and put constraints on the parameters κ_1, κ_2.

Solution II: $(f \pm g)$-type. Next we present our solution II. It is obtained by taking the following ansatz for E and C whose dependence in f, g are of the form $f \pm g$:

$$E = E(f+g)\,, \qquad C = C(f-g)\,. \tag{2.44}$$

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Equation (2.23) then gives

$$C = \gamma(f-g)\,, \tag{2.45}$$

for some constant γ, and (2.24) can be solved by

$$E = \log\left[\frac{\alpha\gamma^2}{8\delta c_1^2}(f+g)\cosh^2\left(c_1\log\frac{c_2}{f+g}\right)\right], \tag{2.46}$$

where c_1, c_2 are integration constants. Without loss of generality one can take $c_1, c_2 > 0$. One can then integrate (2.26) and (2.27) to get

$$\begin{aligned} S =&\; b_1\log(1-2f)(1+2g) + b_2\log(1+2f)(1-2g) + \left(b_3 - 1 + \delta + \frac{\eta}{\alpha}\right)\log(f+g) + \\ &+\frac{2\kappa_1\kappa_2}{\alpha}\log\left(\frac{1}{2}+2fg+2pqrw\right) + \frac{\delta}{\alpha}\left[(1+4c_1^2)\log c_2 - 4\log\cosh\left(c_1\log\frac{c_2}{f+g}\right)\right] \end{aligned} \tag{2.47}$$

and in this case,

$$b_1 = \frac{\kappa_1^2}{\alpha}\,, \qquad b_2 = \frac{\kappa_2^2}{\alpha}\,, \qquad b_3 = 1 - \delta - \frac{\delta(1+4c_1^2) + (\kappa_1+\kappa_2)^2}{\alpha} - \frac{\eta}{\alpha}\,. \tag{2.48}$$

Finally we obtain

$$\begin{aligned} e^{-M} =&\; a_0 f_u g_v[(1-2f)(1+2g)]^{-b_1}[(1+2f)(1-2g)]^{-b_2}(f+g)^{-b_3}\cosh^{a_4}\left(c_1\log\frac{c_2}{f+g}\right)\times \\ &\times\left[\frac{1}{2}+2fg+2pqrw\right]^{-\frac{2\kappa_1\kappa_2}{\alpha}}\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{a\eta}{\alpha}}, \end{aligned} \tag{2.49}$$

where

$$a_0 = c_2^{-\frac{\delta}{\alpha}(1+4c_1^2)}\left(\frac{\alpha\gamma^2}{8\delta c_1^2}\right)^{\frac{\eta}{\alpha}}, \qquad a_4 = \frac{4\delta}{\alpha} + \frac{2\eta}{\alpha}\,. \tag{2.50}$$

The other components of the metric are given by

$$e^{nA} = \left(\frac{\alpha\gamma^2}{8\delta c_1^2}\right)^{\frac{1}{\alpha}}(f+g)^{1+\frac{1}{\alpha}}\cosh^{\frac{2}{\alpha}}\left(c_1\log\frac{c_2}{f+g}\right)\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{a}{\alpha}}, \tag{2.51}$$

$$e^{mB} = \left(\frac{\alpha\gamma^2}{8\delta c_1^2}\right)^{-\frac{1}{\alpha}}(f+g)^{1-\frac{1}{\alpha}}\cosh^{-\frac{2}{\alpha}}\left(c_1\log\frac{c_2}{f+g}\right)\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{-\frac{a}{\alpha}}, \tag{2.52}$$

and the dilaton field is given by

$$e^{\phi} = \left(\frac{\alpha\gamma^2}{8\delta c_1^2}\right)^{-\frac{\delta a}{\alpha}}(f+g)^{-\frac{\delta a}{\alpha}}\cosh^{-\frac{2\delta a}{\alpha}}\left(c_1\log\frac{c_2}{f+g}\right)\left[\left(\frac{w-p}{w+p}\right)^{\kappa_1}\left(\frac{r-q}{r+q}\right)^{\kappa_2}\right]^{\frac{1}{\alpha}}. \tag{2.53}$$

The above (2.31), (2.45)–(2.53) give a five-parameters family of solution in the F-region. As in the $(pqrw)$-type solution given above, they still solve the differential equations in the L-, R- and in the P-region if we take (2.41), (2.42) and (2.43). Patching of the solutions together will put constraints on the parameters $\gamma, c_1, c_2, \kappa_1, \kappa_2$.

3. Junction conditions of metric

In the last section, we have solved the equation of motion in the different regions, and one need to paste together these solutions across the boundaries. To be a physically acceptable solution, it is necessary that the metric $g_{\mu\nu}$ (spacetime) must be continuous and invertible. What about the derivatives of the metric? what about the $R_{\mu\nu}$, $R_{\mu\nu\alpha\beta}$ and the various curvature invariants R, R_2, R_4 etc.? What kind of conditions shall we impose on them?[2]

It is natural to allow the Ricci tensor to be P.C.. This is particularly clear in the presence of a form flux. In this case, the stress tensor does not need to be continuous in spacetime and can generally be P.C. with jump across the junctions. Let $\mathcal{S}$ be a hypersurface where the form flux $F_{[n+1]}$ is discontinuous across. It can be shown that $\mathcal{S}$ has to be a null surface. See the appendix B. Through Einstein equation, this implies that $R_{\mu\nu}$ can have jump across $\mathcal{S}$ too.

It is also natural to assume that the energy momentum tensor does not suffer anything more than a 'shock' discontinuity, i.e. no delta function singularity. Therefore a first requirement on $R_{\mu\nu}$ is that it should contain no delta function singularity.

To achieve this, one may demand the metric to be C^1, and at least piecewise C^2. This is the Lichnerowicz condition. While this is sufficient, it is not necessary. As it turns out, the absence of delta function singularity in $R_{\mu\nu}$ is also possible for metric that is piecewise C^1 and that satisfy a certain special condition. Generally, a piecewise C^1 metric induces a Dirac delta function singularity for $R_{\mu\nu\alpha\beta}$ and in $R_{\mu\nu}$ [23, 17], see (B.6) and (B.7). Such singularities in $R_{\mu\nu}$ is not acceptable and should be killed. As we elucidate in the appendix B, the condition for the absence of the Dirac delta function is precisely the O'Brien-Synge (OS) junction conditions. Let $\mathcal{S}$ be a null surface and let it be defined by $x^0 = \text{const.}$, the OS junction conditions require that

$$g_{\mu\nu}\,, \quad \sum_{i,j} g^{ij} g_{ij,0}\,, \quad \sum_i g^{i0} g_{ij,0}\,, \qquad (i,j \neq 0) \tag{3.2}$$

be continuous across $\mathcal{S}$.

The OS junction condition kills the δ-function singularity in $R_{\mu\nu}$ and leaves us with a P.C. $R_{\mu\nu}$. This is in perfect consistence with the original assumption that the stress tensor is P.C.. The invertibility and continuity of the metric and the piecewise continuity of $R_{\mu\nu}$ implies automatically that R and R_2 are P.C. also. Having the δ-function killed, however, there is still the possibility that R, R_2 may blow up at the junctions, i.e. of the form $\sim u^{-a}, v^{-a}, a > 0$. This would be physically unacceptable and more conditions may have to be imposed on the metric so that this does not appear.

[2]We will distinguish and analysis the behaviour of various quantities (e.g. the first derivatives of the metric, $R_{\mu\nu}$, $R_{\mu\nu\alpha\beta}$ etc.) at the junction. In general a physical quantity can take the following form across a junction, say $u = 0$,

$$h(u) = h^{(0)}(u) + h^{(1)}(u)\Theta(u) + h^{(2)}(u)\delta(u)\,, \tag{3.1}$$

where $h^{(i)}(u)$ are continuous functions for $u > 0$ and $u < 0$, and $h^{(0)}(u)$ is continuous across $u = 0$. The quantity h is continuous if there is no Θ or δ-function, i.e. $h^{(1)}(0) = h^{(2)}(0) = 0$. h is P.C. if $h^{(2)}(0) = 0$ and $h^{(1)}(0) \neq 0$. Note that we allow the jump $h^{(1)}(0)$ to be both finite or infinite. When $h^{(2)}(0) \neq 0$, we have a Dirac delta function singularity at the junction.

As for the δ-function in $R_{\mu\nu\alpha\beta}$ (also possibly in R_4 or other higher curvature invariants), it has been proposed [23] to identify these discontinuity with the impulsive gravitational wavefront. We will give an explicit proof of this in our appendix E. They are therefore physically acceptable. However, it is also possible to impose further conditions on the metric such that $R_{\mu\nu\alpha\beta}$ and R_4 has no δ-function singularity. Similarly, one may impose further conditions such that $R_{\mu\nu\alpha\beta}$ and R_4 has no poles.

Summarizing our discussion above, we conclude that in order to paste together the solutions obtained from solving the differential equation of motion in the different regions, one need to impose the following junction condition on the metric:

1. If the metric is C^1, then impose the Lichnerowicz condition. Otherwise, if the metric is piecewise C^1, then impose the OS junction conditions.

2. Curvature invariants R, R_2 do not blow up at the junction.

In special circumstances, one may also require that

3*. $R_{\mu\nu\alpha\beta}$ and R_4 have no δ-function singularity or blow up at the junction.

4. Colliding plane wave solutions with flux

We now apply the junction conditions to the flux-CPW solutions we find in section 2 and use them to constraint the parameters appearing in the solution. Let us write the near-junction expansion of $f(u \geq 0)$ and $g(v \geq 0)$ as follows:

$$f = f_0(1 - d_1 u^{n_1}), \qquad u \sim 0^+, \tag{4.1}$$

$$g = g_0(1 - d_2 v^{n_2}), \qquad v \sim 0^+. \tag{4.2}$$

In particular the boundary conditions will put restrictions on the *boundary exponents* n_i.

4.1 Imposing junction conditions

1. Lichnerowicz/O'Brien-Synge junction conditions. First we require the metric to be continuous across the junctions. Continuity of e^A and e^B is automatic. If one fixes the normalization of the metric such that $A = B = M = 0$ in the P-region, then we get $f_0 = g_0 = 1/2$. As for the continuity of e^{-M}, the condition (2.41) requires

$$b_1 = 1 - \frac{1}{n_1}, \qquad d_1 = \left(\frac{2}{n_1}\right)^{n_1}. \tag{4.3}$$

As for the condition from (2.42), we just have to replace the subscript 1 by 2 in (4.3).

For the solution II, there are additional constraints to continue the metric in the L/R-region to the flat metric in the P-region, say e^{-M}, e^A and e^B normalized to 1, this requires

$$a_0 \cosh^{a_4}(c_1 \log c_2) = 1, \qquad \frac{\alpha\gamma^2}{8\delta c_1^2}\cosh^2(c_1 \log c_2) = 1. \tag{4.4}$$

These can be simplified to

$$c_2^{\frac{1+4c_1^2}{2}} = \cosh^2(c_1 \log c_2) = \frac{8\delta c_1^2}{\alpha\gamma^2}\,. \tag{4.5}$$

One can solve these constraints for $c_1, c_2 > 0$ in terms of γ

$$c_1 = \sqrt{\frac{\alpha\gamma^2}{8\delta}}\,, \qquad c_2 = 1\,. \tag{4.6}$$

Therefore we get a three-parameters family of solutions depending on γ, κ_1 and κ_2.

Next we ask when the metric is piecewise C^1. Let us consider the junction $u = 0$ in details. The analysis for the $v = 0$ junction is exactly the same. First we claim that for our solution I and II, we have for $u \sim 0$

$$U_u = \left(u^{n_1-1}\frac{-d_1 n_1}{1+2g} + \text{l.s.t.}\right)\Theta(u)\,, \tag{4.7}$$

$$\alpha V_u = \left(u^{\frac{n_1}{2}-1}\cdot \alpha e_1(v) + \text{l.s.t.}\right)\Theta(u)\,, \tag{4.8}$$

$$n A_u = \left(u^{\frac{n_1}{2}-1} e_1(v) + \text{l.s.t.}\right)\Theta(u)\,, \tag{4.9}$$

$$m B_u = \left(-u^{\frac{n_1}{2}-1} e_1(v) + \text{l.s.t.}\right)\Theta(u)\,, \tag{4.10}$$

$$M_u = \left(\kappa_1\kappa_2 u^{\frac{n_1}{2}-1} e_0(v) - \eta u^{\frac{n_1}{2}-1} e_1(v) + \text{l.s.t.}\right)\Theta(u)\,, \tag{4.11}$$

where l.s.t. in the above stands for less singular terms and $e_0(v), e_1(v)$ are some nonzero functions of v. The proof can be found in the appendix D. As a result of (4.9)–(4.11), we find that the metric is C^1 if $n_1 > 2$ and is piecewise C^1 if $n_1 \le 2$.

For the case that metric is C^1, it is easy to see from (4.9)–(4.11) that it is also at least piecewise C^2. Thus the Lichnerowicz condition is satisfied. As for the case that the metric is piecewise C^1, i.e. $n_1 \le 2$, we need to impose the second and third OS junction conditions which require that U_u to be continuous (i.e. equal to zero) across the junction at $u = 0$. From (4.7), it is easy to see the piecewise C^1 metric also satisfies the OS condition only if

$$1 < n_i \le 2\,. \tag{4.12}$$

Alternatively, we can understand why only U_u is constrained by the OS condition from the following fact: the $(n+1)$-th order differential equation is solved provided that the boundary conditions from the zeroth to the n-th derivatives are given. Looking into EOM, we note that the only term of the second derivative with respect to u is U_{uu}, the others are all terms of the first derivative with respect to u. Therefore, besides the continuity of the metric at junction we need to impose the continuity only on U_u but not on M_u, V_u, C_u and ϕ_u. Similarly for the junction condition at $v = 0$.

In summary, from imposing the Lichnerowicz or the O'Brien-Synge junction conditions, we have the following allowed possibilities

$$\begin{cases} \text{(i) } 1 < n_i \le 2: & \text{metric is piecewise } C^1 \\ \text{(ii) } n_i > 2: & \text{metric is at least piecewise } C^2, \end{cases} \tag{4.13}$$

in our solution I (2.31), (2.33)–(2.40) and our solution II (2.31), (2.45)–(2.53).

2. On R, R_2. Having imposed the above junction conditions, the Ricci tensor $R_{\mu\nu}$ is at least P.C.. However it may still blow up at the junction. Now we claim that in order for the curvature invariants R *and* R_2 not to blow up at the junction, we need the condition $n_i \geq 2$ in addition to the junction condition imposed above. To see this, we start with

$$R = 2e^{M} R_{uv} + ne^{-A} R_{xx} + me^{-B} R_{yy} \,, \tag{4.14}$$

where R_{uv} etc. are given in appendix A. We note that only first order derivatives of M, A, B with respect to u or v appear in R_{uv}, R_{xx} and R_{yy} and there is no second order derivatives like A_{uu}, therefore the singularity behavior of R is controlled by U_u, V_u, M_u. Now V_u is the most singular object and from (4.8), it is

$$V_u \sim u^{\frac{n_1}{2}-1} \,, \tag{4.15}$$

therefore R is non-singular at the boundary if $n_i \geq 2$. However this condition may be too strong since there could be cancellation among different pieces in R. To check whether this is the case, one can either examine (4.14) explicitly. Alternatively we can resort to the Einstein equations and obtains the following simple form for R

$$R = 2e^{M} \phi_u \phi_v + \frac{m-n}{m+n} \frac{e^{M-E}}{f+g} C_u C_v \,. \tag{4.16}$$

Using the boundary behaviour (D.13) and (D.14) for the flux solution, we see that for R not to blow up, one requires that $n_1 \geq 2$.

Next we consider R_2. It is

$$R_2 = 2e^{2M} R_{uv}^2 + 2e^{2M} R_{uu} R_{vv} + ne^{-2A} R_{xx}^2 + me^{-2B} R_{yy}^2 \,. \tag{4.17}$$

Resorting to the Einstein equations as we did before for R, we find that R_2 involves only ϕ_u and C_u as R does. One can easily see that R_2 does not blow up at the junction if the condition $n_i \geq 2$ is satisfied.

Taking into account together with the junction condition (4.13), we find the following physical possibilities:

$$b_i = 1 - \frac{1}{n_i} \tag{4.18}$$

$$\begin{cases} \text{(i) } n_i = 2 : & \text{metric is piecewise } C^1 \\ \text{(ii) } n_i > 2 : & \text{metric is at least piecewise } C^2, \end{cases} \tag{4.19}$$

on our solution I (2.31), (2.33)–(2.40), and on our solution II (2.31), (2.45)–(2.53).

3*. On $R_{\mu\nu\alpha\beta}$ and R_4. If one want, one may further restrict the solution so that $R_{\mu\nu\alpha\beta}$ and R_4 have no δ-function singularity and do not blow up at the junction. Let us examine first the δ-function singularity, which is possible when the metric is piecewise C^1. We claim that for the δ-function singularity in $R_{\mu\nu\alpha\beta}$ or R_4 to disappear, it is necessary that $n_i > 2$. Thus in view of (4.19), all the piecewise C^1 metric of (i) results in a δ-function singularity in $R_{\mu\nu\alpha\beta}$ and R_4.

We have two ways to find out the condition for the absence of the δ-function singularity at junction. For the general cases one can use the result in appendix B by requiring

$$D_{u\alpha u\beta} \equiv -h_{\alpha\beta} = \Delta(g_{\alpha\beta,u}) \tag{4.20}$$

and

$$I = -4e^{2M}[n\Delta(A_u)e^{-A}R_{vxvx} + m\Delta(B_u)e^{-B}R_{vyvy}] \,. \tag{4.21}$$

to be vanishing at $u = 0$.

For our particular flux-CPW solution I and II, it is more straightforward to use (4.9)–(4.10) and (A.16) to write down the most singular term in R_4, and the result is

$$R_4 \sim \left(u^{\frac{n_1}{2}-1}\delta(u) + \left(\frac{n_1}{2}-1\right)u^{\frac{n_1}{2}-2}\Theta(u)\right)e_1(v)(A_{vv} - B_{vv}) + \mathcal{O}(u^{n_1-2}\Theta(u)) \tag{4.22}$$

where the sub-leading term $u^{n_1-2}\Theta(u)$ comes from the terms such as A_u^2 and B_u^2 omitted in (A.16). From (4.22) we can summarize the singularity structure in table 1.

Note that the $n_1 = 2$ case has the sharp δ-function plus Θ-function profile which corresponds to an impulsive wavefront with a tail, and the $n_1 = 4$ case has the Θ-function profile, thus a shock wavefront. For $n_1 > 4$ one has smooth wavefront.

	$n_i = 2$	$2 < n_i < 4$	$n_i = 4$	$n_i > 4$
Dirac-δ	yes	no	no	no
step	yes	yes	yes	no
pole	no	yes	no	no

Table 1: Summary of the singularity structure of R_4.

Finally, we would like to comment on the necessity of imposing pole-free condition on the curvature invariants. The pole of the curvature invariants is considered to be problematic because the general relativity break downs there, for example, the black hole singularity or the cosmological singularity at the big bang. On the other hand, the Dirac delta function or Theta function singularities are generally accepted and regarded as an idealized limiting case of localized matter source, for example, in the Khan-Penrose CPW and the shock wave considered in [13]. Based on these, the case $2 < n_i < 4$ in the table I. cannot be accepted. However, to allow for broader classes of CPW solutions, we will not be so restrictive in our discussions. Instead only the condition (4.13) is imposed to ensure that R and R_2 is not blowing up at the junction. However it is straightforward to also require that R_4 not to blow up by rejecting the solutions with $2 < n_i < 4$.

4.2 Physical flux-CPW solution

Now we apply the physical conditions (4.18), (4.19) to the two solutions we obtained in section 2. Recall that

$$b_i = \frac{\kappa_i^2 + \epsilon\delta}{1 + a^2\delta}\,, \qquad \delta = \frac{m+n}{4mn}\,, \tag{4.23}$$

where $\epsilon = 1$ for the first type of solution and $\epsilon = 0$ for the second type of solution. We have the following window for κ_1^2 and κ_2^2,

$$\frac{1}{2} + \delta\left(\frac{a^2}{2} - \epsilon\right) \leq \kappa_1^2, \kappa_2^2 < 1 + \delta(a^2 - \epsilon)\,. \tag{4.24}$$

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Here, the metric is piecewise C^1 when the equality sign holds, and in this case the metric satisfies the OS junction condition. Otherwise the metric is piecewise C^1 and satisfies the Lichnerowicz junction condition.

To discuss further, let us divide the allowed solutions into non-dilatonic and dilatonic cases.

- Case 1. If there is no dilaton so that $a = 0$, $\kappa_1 = \kappa_2 = 0$, then $b_i = \epsilon\delta$.

  In this case, using $\delta \leq 1/2$, it is easy to see that only $m = n = 1$ and $a = 0$ (i.e. no dilaton) of the type-I solution is allowed. And the metric is piecewise C^1 with $n_i = 2$. This is precisely the original case of Bell-Szekeres [17]. Note that from table I, this solution is impulsive, i.e. has a δ-function profile in $R_{\mu\nu\alpha\beta}$ and R_4.

  The higher dimensional generalization of the 4-dimensional Bell-Szekeres solution with a n-form potential (by which we mean $m = n$ and non-dilatonic) has been considered recently by Gutperle and Pioline in [21]. They found for their solution $n_1 = 2n/(2n-1) < 2$ for $n > 1$ and that R_2 blows up at the junction. This is consistent with what our analysis.

  The higher dimensional Einstein-Maxwell CPW solutions (i.e. $a = 0$ and $m > n = 1$) have been considered in [20]. In terms of our notation, their solution are our type-I solution with $4/3 < n_i = 4m/(3m-1) \leq 8/5$ since $m \geq 2$. Therefore their metric is piecewise C^1 and satisfies the OS condition. However R and R_2 blows up.

- Case 2. There is a dilaton profile.

  Note that the bound at the l.h.s. of (4.24) is nonnegative and that the size of the (κ_1, κ_2) window does not depend on the flux amplitude γ in the solution II. We thus have a 2 (or 3) parameters family of solutions labeled by κ_1, κ_2 (and γ). A 4-dimensional solution has been considered by Gurses and Sermutlu in [19] where $m = n = 1$, $\epsilon = 1$, and $|\kappa_1| = |\kappa_2| = a/2$, so that $n_1 = n_2 = 2$. The metric of this solution is piecewise C^1. Our type-I solutions give generalization of this solution. Moreover our type-II solution is completely new and has never appeared in the literature.

4.3 Future singularity of the solution

As known that in 4-dimensional spacetime the future curvature singularity is a general outcome of the collisions of the gravitational plane waves even with arbitrarily small density [6]. It is then curious to see if the future curvature singularity will also generically appear in our new higher dimensional flux-CPW solutions. We will investigate this issue in this subsection.

We first note that the metric may blow up or vanish at

$$f(u) + g(v) = 0\,, \tag{4.25}$$

which define a hypersurface $\mathcal{S}_0$. Near $\mathcal{S}_0$ we then have

$$e^E \sim (f+g)^{-1}\,, \qquad e^X \sim (f+g)^{\kappa_1+\kappa_2}\,, \qquad \cosh\left(c_1 \log\frac{c_2}{f+g}\right) \sim (f+g)^{-c_1}\,. \tag{4.26}$$

This results in the following singular behavior near $f+g=0$

$$
\begin{aligned}
e^{-M} &\sim (f+g)^{-b_3-\frac{\eta}{\alpha}(1-a(\kappa_1+\kappa_2))+(1-\epsilon)(\frac{\eta}{\alpha}-a_4c_1)}\,, && (4.27)\\
e^{nA} &\sim (f+g)^{1-\frac{1}{\alpha}(1-a(\kappa_1+\kappa_2))+(1-\epsilon)(\frac{2}{\alpha})(1-c_1)}\,, && (4.28)\\
e^{mB} &\sim (f+g)^{1+\frac{1}{\alpha}(1-a(\kappa_1+\kappa_2))-(1-\epsilon)(\frac{2}{\alpha})(1-c_1)}\,, && (4.29)\\
e^{\phi} &\sim (f+g)^{\frac{1}{\alpha}(\delta a+\kappa_1+\kappa_2)-(1-\epsilon)(\frac{2\delta a}{\alpha})(1-c_1)}\,, && (4.30)\\
C_u, C_v &\sim 1\,, && (4.31)
\end{aligned}
$$

where $\epsilon = 1$ (resp. 0) for the type-I (resp. -II) solution as before. The regularity and the invertibility of the metric require that the exponents in (4.27), (4.28), (4.29) vanish. However, it is easy to see that it is impossible for the exponents of (4.28) and (4.29) to vanish at the same time. Therefore we conclude that at $f+g=0$ the metric is singular, i.e. either blows up or vanishes. In particular we have a Killing horizon when e^A or e^B vanishes. For example, the Bell-Szekeres solution has a Killing horizon at $f+g=0$.

The above metric singularity could be just a coordinate singularity if the curvature invariants do not blow up on $\mathcal{S}_0$. Next we check the curvature singularity which may appear in R , R_2 and R_4. First we note that, from (4.27)–(4.29) we have

$$
\frac{\partial^{\ell_1+\ell_2}M}{\partial u^{\ell_1}\partial v^{\ell_2}} \sim \frac{\partial^{\ell_1+\ell_2}A}{\partial u^{\ell_1}\partial v^{\ell_2}} \sim \frac{\partial^{\ell_1+\ell_2}B}{\partial u^{\ell_1}\partial v^{\ell_2}} \sim (f+g)^{-(\ell_1+\ell_2)} \qquad (4.32)
$$

near $\mathcal{S}_0$. Then from the expressions of the Ricci tensor and Riemann tensor listed in the appendix A, it is easy to check that the most singular terms near $\mathcal{S}_0$ in R^2, R_2 and R_4 are all taking the following generic form[3]

$$
e^{2M}(f+g)^{-4} \sim \left[(f+g)^{b_3+\frac{\eta}{\alpha}(1-a(\kappa_1+\kappa_2))-(1-\epsilon)(\frac{\eta}{\alpha}-a_4c_1)-2}\right]^2. \qquad (4.33)
$$

Therefore, to avoid the future curvature singularity on $\mathcal{S}_0$ it is required that the exponent in (4.33) to be non-negative, or equivalently

$$
(\kappa_1+\kappa_2)^2+\eta a(\kappa_1+\kappa_2)+\delta-\eta+\alpha(1+\delta) \le -2(1-\epsilon)(1-c_1)(\eta-2\delta c_1)\,, \qquad (4.34)
$$

where we recall that

$$
b_3 = 1-\delta-\frac{\delta+(\kappa_1+\kappa_2)^2}{\alpha}-(1-\epsilon)\left(\frac{\eta+4\delta c_1^2}{\alpha}\right), \qquad a_4 = \frac{4\delta}{\alpha}+\frac{2\eta}{\alpha} \qquad (4.35)
$$

and

$$
\delta = \frac{n+m}{4mn}\,, \qquad \alpha = 1+a^2\delta\,, \qquad \eta = \frac{m-n}{2mn}\,. \qquad (4.36)
$$

Note that the condition (4.34) makes the metric component e^{-M} blows up on $\mathcal{S}_0$.

[3]This is not the case when there is no dilaton. In this case the only solution is the Bell-Szekeres solution and there $R = R_2 = 0$ and $R_4 = const.$ As mentioned above, the singularity at $\mathcal{S}_0$ is not a curvature singularity, but that of a Killing horizon. However when the dilaton is tuned on, one can easily check that the coefficient of (4.33) in R^2, R_2 and R_4 are nontrivial function of, say u, on the hypersurface $\mathcal{S}_0$.

<table>
<tr><th colspan="5">Summary of the CPW solutions with $(n+1)$-form flux in $m+n+2$ dimensions</th></tr>
<tr><td></td><td colspan="2">without dilaton</td><td colspan="2">with nonzero dilaton</td></tr>
<tr><td></td><td>Soln. I ($\epsilon = 1$)</td><td>Soln. II ($\epsilon = 0$)</td><td>Soln. I ($\epsilon = 1$)</td><td>Soln. II ($\epsilon = 0$)</td></tr>
<tr><td>$m = n$</td><td>only $m = n = 1$, i.e., Bell-Szekeres</td><td>no physical solution</td><td rowspan="2">(2.31), (2.33)–(2.40) parameterized by κ_1, κ_2 subjected to (4.24)</td><td rowspan="2">(2.31), (2.45)–(2.53) parameterized by κ_1, κ_2, γ subjected to (4.24)</td></tr>
<tr><td>$m \neq n$</td><td>no physical solution</td><td>no physical solution</td></tr>
</table>

Table 2: Physical flux-CPW solutions.

The condition (4.34) is a complicated one, and in general it can be violated so that a curvature singularity will develop at the late time. To see this, we first note that the l.h.s. of (4.34) is always greater than 1,

$$\text{l.h.s.} = \left(\kappa_1 + \kappa_2 + \frac{\eta a}{2}\right)^2 + 1 + a^2\delta + (2\delta - \eta) + a^2\left(\delta^2 - \frac{\eta^2}{4}\right) > 1 \qquad (4.37)$$

for any κ_1 and κ_2. Therefore type-I solution will always develop late time curvature singularity. As for type-II solution, we note that the r.h.s. of (4.34) is

$$\text{r.h.s.} = -4\delta\left(c_1 - \frac{1}{4n\delta}\right)^2 + \frac{n}{m(n+m)} < 1\,. \qquad (4.38)$$

Thus (4.34) can never be satisfied and curvature singularity will always develop. Therefore, in conclusion, we find that both the type-I and type-II solutions will develop curvature singularity in the future hypersurface $f + g = 0$.

5. Conclusions and discussions

In this paper we have constructed physical solutions of CPW in string theory with non-zero form flux. These solutions solve the EOM in the interior of each region. Moreover: (1) they satisfy the Lichnerowicz/O'Brien-Synge junction conditions, and (2) the curvature invariants R, R_2 do not blow up at the junctions. The results of this paper can be summarized in the table.

In this paper, we have obtained two types of solution using two different ansatz. The solution I follows the ansatz of the original Bell-Szekeres solution. The solution II is new and has never appeared in the literature before. We see that the original Bell-Szekeres solution cannot be generalized to higher dimensions unless the dilaton field is turned on. Roughly speaking, a particular combination (2.20) of the dilaton field and the metric component V is given by (2.31) and behaves like the V in the pure CPW. As we explained before, this field X makes an important contribution to the metric, and when suitably restricted (see (4.24)) could smoothen out the singularities in the solution and lead to physically acceptable CPW solution.

We have also shown that all the solutions (except for the Bell-Szekeres solution as explained before) in the table II. will result in a late time curvature singularity on a hypersurface $\mathcal{S}_0$ defined by (4.25). In [6] Tipler showed that the collisions of 4-dimensional gravitational plane waves will develop a late time curvature singularity due to the artificial plane symmetry. Our results support the generalized version of Tipler's theorem in higher dimensional supergravity.

One of our original motivation in this project was to construct flux-CPW in M-theory since the spectrum of fields are very simple. However we failed to obtain such solution, both for the 11 dimensional and for the 27 dimensional M-theory [22]. In retrospect, this is understandable since there is no dilaton in these theories. A more general ansatz than those we considered may be needed. So far only purely gravitational CPW solution has been constructed in these theories. One may wonder whether the absence of flux-CPW solution in these theories has any fundamental meaning? It is important to try to construct flux-CPW solutions for these non-dilatonic theories.

It may be interesting to consider CPW with more than one higher form field turning on. For example, one may consider having a $F_{[n+1]}$ field strength together with a a $F_{[m+1]}$ field strength simultaneously. This will be very easy to deal with using our ansatz (2.5) for the metric. A more general metric ansatz will be needed when the potential do not have complementary dimensions. Similarly, it may be interesting to study the scattering of waves with both the dilaton and axion turned on in IIB string theory.

Finally, we hope that the physical flux-CPW solutions constructed in this paper will shed new lights or find applications on the issues of cosmological singularity or its resolution in the context of string theory.

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A. Ricci and Riemann tensors

For the metric

$$ds^2 = 2e^{-M}dudv + e^A \sum_{i=1}^{n} dx_i^2 + e^B \sum_{j=1}^{m} dy_j^2 \tag{A.1}$$

with the functions M, A, B being functions of u, v. We have

$$R_{uu} = -\frac{1}{2}\left[nA_{uu} + mB_{uu} + nM_uA_u + mM_uB_u + \frac{1}{2}(nA_u^2 + mB_u^2)\right], \tag{A.2}$$

$$R_{vv} = -\frac{1}{2}\left[nA_{vv} + mB_{vv} + nM_vA_v + mM_vB_v + \frac{1}{2}(nA_v^2 + mB_v^2)\right], \tag{A.3}$$

$$R_{uv} = M_{uv} - \frac{n}{2}A_{uv} - \frac{m}{2}B_{uv} - \frac{1}{4}(nA_uA_v + mB_uB_v), \tag{A.4}$$

$$R_{xx} = -\frac{1}{2}e^{M+A}\left[2A_{uv} + nA_uA_v + \frac{m}{2}(A_uB_v + A_vB_u)\right], \tag{A.5}$$

$$R_{yy} = -\frac{1}{2}e^{M+B}[2B_{uv} + mB_uB_v + \frac{n}{2}(A_uB_v + A_vB_u)]. \tag{A.6}$$

Here $x = x_i$ for $i = 1, \ldots, n$ and $y = y_j$ for $j = 1, \ldots, m$.

In the following we list the independent nonvanishing components of the Riemann tensor for the metric (A.1):

$$R_{uvuv} = -e^{-M}M_{uv}, \tag{A.7}$$

$$R_{xyxy} = \frac{-1}{4}e^{M+A+B}(A_uB_v + A_vB_u), \tag{A.8}$$

$$R_{uxvx} = -e^{A}\left(\frac{1}{2}A_{uv} + \frac{1}{4}A_uA_v\right), \tag{A.9}$$

$$R_{vxvx} = -e^{A}\left(\frac{1}{2}A_{vv} + \frac{1}{2}M_vA_v + \frac{1}{4}A_v^2\right), \tag{A.10}$$

$$R_{uxux} = -e^{A}\left(\frac{1}{2}A_{uu} + \frac{1}{2}M_uA_u + \frac{1}{4}A_u^2\right), \tag{A.11}$$

$$R_{uyvy} = -e^{B}\left(\frac{1}{2}B_{uv} + \frac{1}{4}B_uB_v\right), \tag{A.12}$$

$$R_{vyvy} = -e^{B}\left(\frac{1}{2}B_{vv} + \frac{1}{2}M_vB_v + \frac{1}{4}B_v^2\right), \tag{A.13}$$

$$R_{uyuy} = -e^{B}\left(\frac{1}{2}B_{uu} + \frac{1}{2}M_uB_u + \frac{1}{4}B_u^2\right). \tag{A.14}$$

As usual

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{[\rho\sigma][\mu\nu]}. \tag{A.15}$$

Moreover, we also have that near $u = 0$ or $v = 0$,

$$R_4 = \frac{e^{2M}}{4}(nA_{uu}A_{vv} + mB_{uu}B_{vv}) + \cdots, \tag{A.16}$$

where $\cdots$ denotes terms that contains first order derivatives with respect to u or v and are less singular at the junctions than the second derivative terms A_{uu} etc.

B. On the O'Brien-Synge junction conditions and beyond

As noted by Bell-Szekeres [17], the original Khan-Penrose pure CPW metric is piecewise C^1 and does not satisfy the Lichnerowicz condition. However the Khan-Penrose solution is physical and perfectly acceptable. Moreover, in the case of electromagnetism where there could be different electromagnetic field configurations in different regions of spacetime, it is necessary to allow for metrics that are piecewise C^1 such that the Ricci tensor is P.C. However in general, the Ricci tensor of a piecewise C^1 metric has δ-function singularity. In this appendix, we study the conditions for the absence of such singularities. We will show explicitly that the requirement that *the Ricci tensor to be P.C. is equivalent to the O'Brien-Synge junction conditions on the metric.* Given a null[4] surface $\mathcal{S}$ defined by $x^0 = const.$, the OS junction conditions require that

$$g_{\mu\nu}\,, \quad \sum_{i,j} g^{ij} g_{ij,0}\,, \quad \sum_{i} g^{i0} g_{ij,0}\,, \qquad i,j \neq 0 \tag{B.4}$$

be continuous across $\mathcal{S}$.

To prove this, we first recall that that if the metric $g_{\mu\nu}$ is continuous and $g_{\mu\nu,\rho}$ is P.C. across $\mathcal{S}$ defined by $u(x^\mu) = 0$, then $g_{\mu\nu,\rho}$ must take the form [23]

$$g_{\mu\nu,\rho} = g^{(0)}_{\mu\nu,\rho} + h_{\mu\nu} u_\rho \Theta(u) \tag{B.5}$$

for some $h_{\mu\nu}$ and u_ρ is the normal to $\mathcal{S}$. The piece $g^{(0)}_{\mu\nu,\rho}$ is continuous across $\mathcal{S}$, and $\Theta(u)$ is the Heaviside step function. Furthermore, Bell-Szekeres [17] showed that (B.5) yields

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= R^{(1)}_{\mu\nu\rho\sigma} + 2u_{[\mu}h_{\nu][\rho}u_{\sigma]}\delta(u)\,, \\ g_{\mu\nu,\rho\sigma} &= g^{(1)}_{\mu\nu,\rho\sigma} + h_{\mu\nu}u_\rho u_\sigma \delta(u)\,, \end{aligned} \tag{B.6}$$

where the superscript (1) refers to the piecewise continuity of the quantity across $\mathcal{S}$. It follows immediately that

$$R_{\nu\sigma} = R^{(1)}_{\nu\sigma} + 2N_{\nu\sigma}\delta(u)\,, \qquad \text{where } N_{\nu\sigma} := h_{\mu\nu}u^\mu u_\sigma + h_{\mu\sigma}u^\mu u_\nu - h^\lambda_\lambda u_\nu u_\sigma\,. \tag{B.7}$$

Therefore for the Ricci tensor to be P.C., i.e. the $\delta(u)$ piece vanishes, it is necessary that

$$N_{\nu\sigma} = h_{\mu\nu}u^\mu u_\sigma + h_{\mu\sigma}u^\mu u_\nu - h^\lambda_\lambda u_\nu u_\sigma = 0\,. \tag{B.8}$$

Now we show that the condition (B.8) for $R_{\mu\nu}$ to be P.C. is equivalent to OS conditions mentioned above. Take $u = x^0 = const.$ to be the null surface $\mathcal{S}$. And choose the normal to $\mathcal{S}$ to be $u_\mu = \delta^\mu_0$. That the surface is null implies that $g^{00} = 0$. The condition (B.8) gives the following nontrivial conditions

$$N_{00} = 2h_{\mu 0}g^{\mu 0} - h^\lambda_\lambda = 0\,, \qquad \text{and} \qquad N_{0i} = h_{i\mu}g^{\mu 0} = 0\,. \tag{B.9}$$

[4]We recall here the reason why null surface discontinuity is relevant for the problem of CPW with P.C. higher form field. In general, consider a n-form field strength $F_{[n]}$, and assume that F be P.C. with a discontinuity across a certain hypersurface $\mathcal{S}$ described by the equation $u(x^\mu) = 0$. Then $F_{[n]}$ takes the form

$$F_{\mu_1\mu_2\cdots\mu_n} = f_{\mu_1\mu_2\cdots\mu_n} + \psi_{\mu_1\mu_2\cdots\mu_n}\Theta(u)\,, \tag{B.1}$$

where $f_{\mu_1\mu_2\cdots\mu_n}$ is continuous. Then

$$F_{\mu_1\mu_2\cdots\mu_n,\rho} = f_{\mu_1\mu_2\cdots\mu_n,\rho} + \psi_{\mu_1\mu_2\cdots\mu_n,\rho}\Theta(u) + \psi_{\mu_1\mu_2\cdots\mu_n}u_\rho\delta(u), \tag{B.2}$$

where $u_\rho := \partial_\rho u$ is the normal derivative to $\mathcal{S}$. It follows from the Bianichi identity and the field equation that

$$\psi_{[\mu_1\mu_2\cdots\mu_n}u_{\rho]} = 0\,, \qquad \psi_{\mu_1\mu_2\cdots\mu_{n-1}\rho}u_\rho = 0\,. \tag{B.3}$$

Contracting the first equation with u^ρ and we get $u_\rho u^\rho = 0$. Hence the surface of discontinuity must be null.

Denotes by $\Delta(H)$ the discontinuity of a function H across $\mathcal{S}$, it is $h_{\mu\nu} = \Delta(g_{\mu\nu,0})$. We have

$$N_{00} = 2\Delta(g_{\mu 0,0})g^{\mu 0} - \Delta(g_{\mu\lambda,0})g^{\mu\lambda} = -\Delta(g_{ij,0})g^{ij} \tag{B.10}$$

and

$$N_{0i} = \Delta(g_{ij,0})g^{j0} \,. \tag{B.11}$$

Therefore the condition (B.9) is precisely the same as the OS junction condition. This conclude our proof.

We remark that for our metric ansatz (2.5), the OS junction condition requires that, say, across the junction $u = 0$:

$$U, V, M \text{ continuous}, \qquad \text{and} \qquad U_u \to 0, \quad u \to 0^+ \,. \tag{B.12}$$

Note that there is no requirement on the other normal derivatives V_u or M_u.

Next we examine the singularity in $R_{\mu\nu\rho\sigma}$ and R_4. Let us denote the coefficient of the δ-function singularity in (B.6) by

$$D_{\mu\nu\rho\sigma} := 2u_{[\mu}h_{\nu][\rho}u_{\sigma]} \,. \tag{B.13}$$

Consider the junction $u = x^0 = 0$, it is easy to see that the only independent nonvanishing component of $D_{\mu\nu\rho\sigma}$ is

$$D_{0\alpha 0\beta} = -2h_{\alpha\beta} = -2\Delta(g_{\alpha\beta,0}) \,. \tag{B.14}$$

Thus the δ-function singularity in $R_{\mu\nu\rho\sigma}$ is absent if the normal derivatives of the metric is continuous across the junction.

As for R_4, we note that for the form of our metric ansatz (2.5), it is easy to show that

$$R_4 = R_4^{(1)} + I\delta(u) \,, \qquad I := 4D_{0\alpha 0\beta}R^{0\alpha 0\beta} \tag{B.15}$$

across the $u = 0$ boundary. Here $R_4^{(1)}$ is P.C.. Note that there is no $(\delta(u))^2$ term.

C. CPW solution without flux

In this appendix, we give the pure colliding gravitational and dilatonic plane waves in the higher dimensional dilaton gravity *without* turning on the form flux. Note that both the dilaton ϕ and the metric component V obey the same equation as in the standard purely gravitational case, see (2.13) and (2.18) by setting C and a to zero. Explicitly, the pure dilatonic and gravitational CPW solution is given by

$$\phi = \kappa_1 \log\frac{w-p}{w+p} + \kappa_2 \log\frac{r-q}{r+q} \,, \tag{C.1}$$

$$V = k_1 \log\frac{w-p}{w+p} + k_2 \log\frac{r-q}{r+q} \,, \tag{C.2}$$

$$\begin{aligned} e^{-M} = {} & f_u g_v [(1-2f)(1+2g)]^{-b_1}[(1+2f)(1-2g)]^{-b_2}(f+g)^{-b_3} \times \\ & \times \left[\frac{1}{2} + 2fg + 2pqrw\right]^{-2\kappa_1\kappa_2 - 2\delta k_1 k_2} \left[\left(\frac{w-p}{w+p}\right)^{k_1}\left(\frac{r-q}{r+q}\right)^{k_2}\right]^{\eta} , \end{aligned} \tag{C.3}$$

$$e^{nA} = (f+g)\left(\frac{w-p}{w+p}\right)^{k_1}\left(\frac{r-q}{r+q}\right)^{k_2}, \tag{C.4}$$

$$e^{mB} = (f+g)\left[\left(\frac{w-p}{w+p}\right)^{k_1}\left(\frac{r-q}{r+q}\right)^{k_2}\right]^{-1}, \tag{C.5}$$

where

$$b_1 := \kappa_1^2 + \delta k_1^2\,, \qquad b_2 := \kappa_2^2 + \delta k_2^2\,, \qquad b_3 := 1 - \delta - (\kappa_1+\kappa_2)^2 - \delta(k_1+k_2)^2\,, \tag{C.6}$$

and δ and η are defined as before.

As for the junction condition to be imposed for these solutions, we have already analyzed that one should impose (4.19). This gives

$$\frac{1}{2} \leq \kappa_1^2 + \delta k_1^2,\ \kappa_2^2 + \delta k_2^2 < 1\,, \tag{C.7}$$

where equality holds when the metric is piecewise C^1. The parameters window is quite different from (4.24) for the flux-CPW.

The 4-dimensional Khan-Penrose solution has $n_1 = 2$, the metric is piecewise C^1 and satisfies the OS condition. The higher dimensional generalization of the 4-dimensional Khan-Penrose solution (which we mean a scattering of purely gravitational wave with $m = n$, and where the metric is piecewise C^1) can be obtained by setting $\kappa_1 = \kappa_2 = 0$ in the above, and the $b_i = 1/2$ condition gives

$$k_i^2 = m = n\,. \tag{C.8}$$

This is in contrast to the 4-dimensional Bell-Szekeres solution which has no higher dimensional generalization satisfying the OS junction condition.

Finally we show that singularity always develop in the F-region at the hypersurface $\mathcal{S}_0 : f+g = 0$. As in section 4.3, it is easy to see that the most singular terms near $f+g=0$ in R^2, R_2 and R_4 are all taking the same generic form

$$e^{2M}(f+g)^{-4} \sim [(f+g)^{b_3-\eta(k_1+k_2)-2}]^2\,. \tag{C.9}$$

Future curvature singularity can be avoid if the exponent is non-negative, i.e. if

$$0 \geq 1 + \delta + (\kappa_1+\kappa_2)^2 + \delta(k_1+k_2)^2 + \eta(k_1+k_2)\,. \tag{C.10}$$

However r.h.s. is equal to

$$\text{r.h.s.} = (\kappa_1+\kappa_2)^2 + \delta\left(k_1+k_2+\frac{\eta}{2\delta}\right)^2 + 1 + \frac{1}{\delta}\left(\delta^2 - \frac{\eta^2}{4}\right) \geq 1\,. \tag{C.11}$$

Hence future curvature singularity will always develop.

D. Boundary behaviour of the CPW solution

In this appendix, we will analysis the boundary behaviour of the CPW solution.

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Flux-CPW. We claim that for our flux-CPW solution of type I and type II, we have as $u \sim 0^+$,

$$U_u = \left(u^{n_1-1}\frac{-d_1 n_1}{1+2g} + \text{l.s.t.}\right)\Theta(u)\,, \tag{D.1}$$
$$\alpha V_u = \left(u^{\frac{n_1}{2}-1}\cdot \alpha e_1(v) + \text{l.s.t.}\right)\Theta(u)\,, \tag{D.2}$$
$$nA_u = \left(u^{\frac{n_1}{2}-1} e_1(v) + \text{l.s.t.}\right)\Theta(u)\,, \tag{D.3}$$
$$mB_u = \left(-u^{\frac{n_1}{2}-1} e_1(v) + \text{l.s.t.}\right)\Theta(u)\,, \tag{D.4}$$
$$M_u = \left(\kappa_1\kappa_2 u^{\frac{n_1}{2}-1} e_0(v) - \eta u^{\frac{n_1}{2}-1} e_1(v) + \text{l.s.t.}\right)\Theta(u)\,, \tag{D.5}$$

where l.s.t. in the above stands for less singular terms and $e_0(v), e_1(v)$ are some nonzero functions of v.

To prove this, we start with

$$X_u = \frac{f_u}{f+g}\frac{\kappa_1 wr + \kappa_2 pq}{\sqrt{\frac{1}{4}-f^2}} = -u^{\frac{n_1}{2}-1}\cdot \kappa_1 n_1\sqrt{d_1}\sqrt{\frac{1}{1+2g}} + \text{l.s.t.} \tag{D.6}$$

for $u \sim 0^+$. And for the solution I,

$$E_u^{(\mathrm{I})} = -\frac{f_u}{f+g}\sqrt{\frac{1-4g^2}{1-4f^2}} = u^{\frac{n_1}{2}-1}\cdot n_1\sqrt{\frac{d_1}{2}}\sqrt{\frac{1-2g}{1+2g}} + \text{l.s.t.}\,, \tag{D.7}$$

which has the same singular behavior as X_u; however, for the solution II,

$$E_u^{(\mathrm{II})} = \frac{f_u}{f+g}\left(1 - 2c_1\tanh\left(c_1\log\frac{c_2}{f+g}\right)\right)\,, \tag{D.8}$$

which is less singular than X_u. Combining these together, we have

$$\alpha V_u = E_u + aX_u = u^{\frac{n_1}{2}-1}\cdot \alpha e_1(v) + \text{l.s.t.}\,, \tag{D.9}$$

for some nonzero function $e_1(v)$. Now since

$$U_u = \frac{f_u}{(f+g)} \sim \frac{u^{n_1-1}}{(1/2+g)}\,, \tag{D.10}$$

is less singular compared to V_u. Therefore from (2.14), we have $nA_u \sim -mB_u \sim V_u$, i.e. (4.9), (4.10). As for M_u, we have for $u \sim 0^+$

$$S_u = \frac{2b_1 f_u}{1-2f} + \frac{\kappa_1\kappa_2 n_1\sqrt{d_1/2}}{\alpha}u^{\frac{n_1}{2}-1}\sqrt{\frac{1-2g}{1+2g}} + \text{l.s.t.} = \frac{2b_1 f_u}{1-2f} + \kappa_1\kappa_2 u^{\frac{n_1}{2}-1} e_0(v) + \text{l.s.t.}\,, \tag{D.11}$$

where $e_0(v)$ is some nonzero function. Therefore, from (2.28), (2.35), (2.47), we have the following expansion of M_u

$$\begin{aligned} M_u &= -\frac{f_{uu}}{f_u} + S_u - \eta V_u + \text{l.s.t.} \\ &= -\frac{n_1 - 1 - b_1 n_1}{u} + \kappa_1\kappa_2 u^{\frac{n_1}{2}-1} e_0(v) - \eta u^{\frac{n_1}{2}-1}\cdot e_1(v) + \text{l.s.t.}\,. \end{aligned} \tag{D.12}$$

The first term in (D.12) is zero because $b_1 = 1 - 1/n_1$. And we arrive at (4.11). For convenience, we also note that the singular behavior of $\alpha\phi_u = X_u - a\delta E_u$ is the same as V_u

$$\phi_u \sim u^{\frac{n_1}{2}-1}, \qquad u \sim 0^+ . \tag{D.13}$$

Note also the following behaviour of the flux potential

$$C_u^{(\mathrm{I})} = -\frac{\gamma}{2}(wr+pq)\frac{f_u}{\sqrt{\frac{1}{4}-f^2}} \sim u^{\frac{n_1}{2}-1}, \qquad \text{for the solution I,} \tag{D.14}$$

$$C_u^{(\mathrm{II})} = \gamma f_u \sim u^{n_1-1}, \qquad \text{for the solution II.} \tag{D.15}$$

CPW without flux. For the CPW solution in appendix C, it is easy to see that as $u \sim 0^+$:

$$U_u, V_u, A_u, B_u \sim \left(u^{n_1-1} + \text{l.s.t.}\right)\Theta(u), \tag{D.16}$$

$$M_u = \left((\kappa_1\kappa_2 + \delta k_1 k_2)u^{\frac{n_1}{2}-1}\tilde{e}_0(v) - \eta u^{n_1-1}\tilde{e}_1(v) + \text{l.s.t.}\right)\Theta(u), \tag{D.17}$$

where l.s.t. in the above stands for less singular terms and $\tilde{e}_0(v), \tilde{e}_1(v)$ are some nonzero functions of v.

E. Incoming wave in the Brinkmann coordinates and impulsive wavefront

In the literature, the CPW solution is usually obtained using the metric ansatz of the form (2.5), i.e. in the Rosen coordinates. Sometime it is also useful to rewrite the solution in the Brinkmann coordinates. In this appendix we will show that δ-function singularity appearing in the Riemann tensor $R_{\mu\nu\rho\sigma}$ can be identified with the impulsive component in the wavefront of the wave when written in the Brinkmann coordinates.

In the incoming region, e.g., in the R-region ($v < 0$), the metric in the Brinkmann coordinates takes the form

$$ds^2 = 2dx^+dx^- + \left(H_x(x^+)\sum_{i=1}^{n} X_i^2 + H_y(x^+)\sum_{j=1}^{m} Y_j^2\right)(dx^+)^2 + \sum_{i=1}^{n} dX_i^2 + \sum_{j=1}^{m} dY_j^2, \tag{E.1}$$

where x^+ is related to u through the relation

$$e^{-M}du = dx^+ . \tag{E.2}$$

Note that this relation tells us that x^+ is monotonically increasing with respect to u. Without loss of generality, we can pick $x^+ = 0$ to correspond to $u = 0$. Thus one can replace $\Theta(u)$ by $\Theta(x^+)$ and $\delta(u)$ by $\delta(x^+)$.

The metric in the Brinkmann coordinate is related to that in the Rosen coordinate by

$$H_x = e^{-A}\frac{d^2e^A}{dx^{+2}} = e^{2M}(A_{uu} + M_uA_u + A_u^2), \tag{E.3}$$

$$H_y = e^{-B}\frac{d^2e^B}{dx^{+2}} = e^{2M}(B_{uu} + M_uB_u + B_u^2). \tag{E.4}$$

They contains the similar second derivative terms as in R_4 of (A.16) so that their near junction behaviors are the same as the one of R_4, which is summarized in table I. We then conclude that the singular structure of R_4 at the junction is the same as the wavefront profile of the wave when written in the Brinkmann coordinates.

To be complete and explicit, we can write down the form of the Brinkmann wavefront profile by using the near junction behavior of A_u, B_u and M_u listed in (4.9)–(4.11). Near $u = 0$, we get

$$H_x = \frac{e_1}{n}\left[u^{\frac{n_1}{2}-1}\delta(x^+) + \left(\frac{n_1}{2}-1\right)u^{\frac{n_1}{2}-2}\Theta(x^+) + \frac{e_1}{n}(1-n\eta)u^{n_1-2}\Theta(x^+)\right] + \text{l.s.t.}\,, \quad \text{(E.5)}$$

$$H_y = \frac{e_1}{m}\left[u^{\frac{n_1}{2}-1}\delta(x^+) + \left(\frac{n_1}{2}-1\right)u^{\frac{n_1}{2}-2}\Theta(x^+) + \frac{e_1}{m}(1-m\eta)u^{n_1-2}\Theta(x^+)\right] + \text{l.s.t.}\,, \quad \text{(E.6)}$$

where have set $v = 0$ in (4.9)–(4.11), used the fact $e_0(0) = 0$ and abbreviated $e_1(0) := e_1$. As expected, their singular structures are the same as the one given in (4.22).

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